Comment on ”A class of exact two-dimensional kinetic current sheet equilibria” by P. H. Yoon and A. T. Y. Lui

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Abstract

The analytical derivation of equilibrium solutions described in Yoon and Lui (2005) is discussed.

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In a recent article Yoon and Lui (2005) discuss a class of two-dimensional (2-D) kinetic current sheet equilibria. This class is a 2-D extension of the initial 1-D work of Harris which itself has been extended in different papers (in addition to those presented in Yoon and Lui (2005), see Channell (1976); Mottez (2003); Génot et al. (2005)). This class of equilibria includes results derived by other authors (termed the Harris-Fadeev-Kan-Manankova solution), a generalization of this solution, and new configurations for isolated X-line and isolated magnetic island. In this comment, I want to address an issue related to the analytical derivation of equilibrium solutions. A mean to locate possible singularities of solutions is presented with the aim to complement an aspect of the paper : the requirements with respect to the extent of the smooth domain in typical applications.

The key point of the analytical approach at the heart of the paper is that the general solution for the potential Ψ to the 2-D Grad-Shafranov equation

$$\Delta\Psi = e^{-2\Psi} \tag{1}$$

is obtained in terms of the generating function $g(\zeta)$ with $\zeta = X + iZ$ where X and Z are two dimensionless spatial coordinates (we respect notations used in Yoon and Lui (2005)). The formal solution to the previous equation is given by

$$e^{-2\Psi} = 4|g'|^2/(1+|g|^2)^2 \tag{2}$$

where $g' = dg(\zeta)/d\zeta$. This solution probably appears for the first time in the literature in Liouville (1853). However the generating function $g(\zeta)$ is far from being arbitrary; indeed, it must satisfy in any restricted domain of the complex plane $\mathbb{C}$ the following condition :

$$\Delta \ln |g'| = 0 \tag{3}$$

Let us briefly derive the necessary condition imposed by Equation 3. We suppose the solution of Equation 1 to be of the form :

$$\Psi = -\frac{1}{2}\ln\left(\frac{4|g'|^2}{(1+|g|^2)^2}\right) \tag{4}$$

Taking the Laplacian ($\Delta = \partial^2/\partial X^2 + \partial^2/\partial Z^2 = 4\partial^2/\partial\zeta\partial\bar{\zeta}$) of this expression, we obtain after some elementary algebra :

$$\Delta\Psi = -\Delta\ln|g'| + \Delta\ln(1+|g|^2) = -\Delta\ln|g'| + \frac{4|g'|^2}{(1+|g|^2)^2} \tag{5}$$

Finally, if we impose $\Delta \ln |g'| = 0$, Equation 5 reduces to :

$$\Delta\Psi = \frac{4|g'|^2}{(1+|g|^2)^2} \tag{6}$$

In this case the second term of Equation 5 is exactly $e^{-2\Psi}$ and Equation 1 is exactly solved. More accurately, the condition imposed by Equation 3 expresses the fact that g' must have no zero, pole or essential singularity. Therefore g' has to be an entire function without zero; we recall that a complex function is said to be entire if it is analytic (or holomorphic, to use a more mathematician terminology) at all finite points of the complex plane. This condition is a strict limitation as any function of this type has the general form $\exp(h(\zeta))$ where $h(\zeta)$ is an arbitrary entire function. Thus, in a non restricted domain, Equation 2 gives solutions which increase at infinity and finally are not very interesting from a physical point of view. Two known physical solutions are the Bennett pinch and the periodic pinch solutions (equivalent to Harris solution). They correspond to the particular cases: $h(\zeta) = 0$ and, $h(\zeta) = \zeta$. It is not excluded that other forms for h could be physically interesting. Nevertheless, it seems unlikely that Equation 2 could lead to exact solutions describing localized (non-periodical) structures embedded in the current sheet.

From the derivation above it is clear that the condition expressed by Equation 3 is intrinsically attached to the general form of the solution given by Equation 2. With this in mind, it appears that Equation 3 gives a way to locate the singularities directly from properties of the generating function g; therefore one does not need to find them from the final expression of Ψ which was done in sections 3.3 to 3.9 of the paper. It is particularly convenient when Ψ takes a rather complicated form.

Not satisfying the condition expressed by Equation 3 corresponds to a somewhat different problem. For instance, Tur and Yanosky (2004) considered the case $g'(\zeta) = A(\zeta - \zeta_0)^n$, meaning that g' has nth order zero in point ζ_0. Since

$$\Delta \ln |\zeta - \zeta_0|^n = 2\pi n \delta(\boldsymbol{x} - \boldsymbol{x_0}) \tag{7}$$

(with δ standing for the Dirac distribution function and $\boldsymbol{x_0}$ the singularity coordinate), this particular generating function plugged into Equation 2 gives a potential Ψ which satisfies

$$\Delta\Psi = e^{-2\Psi} - 2\pi n \delta(\boldsymbol{x} - \boldsymbol{x_0}) \tag{8}$$

The solutions of Equation 8 form a class of explicit exact stationary solutions to the 2-D Euler equation which describe vortex patterns in rotational shear flow. This also corresponds to solutions to the Grad-Shafranov equation to which a term representing an external localized current distribution has been added.

To conclude, we note that the limitation imposed by the entire nature of g' is one of the motivations which lead authors to propose asymptotic methods of resolution. This is done in Schindler (1972) in which the analysis applies to the case where plasma boundaries can be determined at once (as in laboratory plasmas). In a free boundary case, as in natural plasmas, an asymptotic theory to find localized structures has been developed in Tur et al. (2001).

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